Risk-Neutral Pricing
Part I - Finding The Arbitrage

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Our economy consists of three assets where the prices of these assets are known at time zero ($t = 0$). There are two possible states-of-the-world at time one ($t = 1$). The two possible states are ω_a and ω_b. From the vantage point of time zero we do not know which state we will be in at time one but we do know the attendant probabilities. The table below presents asset prices at time zero and asset payoffs at time one where the payoff amount depends on the state-of-the-world at time one...

Table 1 - Our Two State Economy

Asset	Price	Payoff t = 1	
Symbol	t = 0	ω_a	ω_b
B	100	105	105
X	80	120	60
Y	50	40	80

The table below presents the time zero probabilities of finding ourselves in either state ω_a or state ω_b at time one...

Table 2 - State Probabilities

Description	Symbol	Probability
Probability that we will find ourselves in state ω_a at time one	p	0.60
Probability that we will find ourselves in state ω_b at time one	$1 - p$	0.40

Our Hypothetical Problem

We will define an arbitrage as the ability to earn a positive return at no risk on a zero investment. An arbitrage portfolio has the following characteristics...

1 Has zero cost to set up
2 Has non-negative values in the future
3 May be of positive value in the future

By creating such a portfolio an investor would receive at no cost the possibility of receiving money in the future.

Question: Given the asset prices and payoffs in Table 1 can we set up an arbitrage such that we invest \$0 at time zero yet receive a payoff of \$100 at time one regardless of the state-of-the-world at that time?

Asset Pricing At Time Zero

Asset B is a risk-free asset in that it's payoff at time one is known with certainty at time zero. The payoff in time one on Asset B is \$105 (B_a) in state ω_a and \$105 (B_b) in state ω_b. Given that the risk-free rate of return in our economy is 5% the equation for the price of Asset B at time zero (B_0) using Tables 1 and 2 above is...

$$B_0 = \frac{B_a\, p + B_b\,(1-p)}{1 + \text{discount rate}} = \frac{(105)(0.60) + (105)(0.40)}{1.05} = 100 \tag{1}$$

Asset X is a risky asset in that it's payoff at time one is not known with certainty at time zero. The payoff in time one on Asset X is either \$120 (X_a) in state ω_a or \$60 (X_b) in state ω_b. Given that the discount rate applicable to this asset is 20% the equation for the price of Asset X at time zero (X_0) using Tables 1 and 2 above is...

$$X_0 = \frac{X_a\,p + X_b\,(1-p)}{1 + \text{discount rate}} = \frac{(120)(0.60) + (60)(0.40)}{1.20} = 80 \tag{2}$$

Asset Y is also a risky asset in that it's payoff at time one is not known with certainty at time zero. The payoff in time one on Asset Y is either \$40 (Y_a) in state ω_a or \$80 (Y_b) in state ω_b. Given that the discount rate applicable to this asset is 12% the equation for the price of Asset Y at time zero (Y_0) using Tables 1 and 2 above is...

$$Y_0 = \frac{Y_a\,p + Y_b\,(1-p)}{1 + \text{discount rate}} = \frac{(40)(0.60) + (80)(0.40)}{1.12} = 50 \tag{3}$$

Finding The Arbitrage

To create the arbitrage portfolio at time zero we will either long or short assets B, X and Y. We therefore make the following definitions in Table 3 below...

Table 3 - Arbitrage Portfolio Composition

Description	Symbol
Units of Asset B that we either long or short at time zero	θ_b
Units of Asset X that we either long or short at time zero	θ_x
Units of Asset Y that we either long or short at time zero	θ_y

If theta is positive then we hold a long position in that asset at time zero and must pay the purchase price (negative cash flow) to acquire that asset. If theta is negative then we hold a short position in that asset at time zero and receive the purchase price (positive cash flow) when we sell that asset short. The equation for the cost of the arbitrage portfolio at time zero is...

$$-\theta_b\,B_0 - \theta_x\,X_0 - \theta_y\,Y_0 = \text{Net cash inflow/(outflow) at time zero} \tag{4}$$

Since we defined an arbitrage as having a zero investment at time zero the equation for the cost of the arbitrage portfolio at time zero becomes...

$$-\theta_b\,B_0 - \theta_x\,X_0 - \theta_y\,Y_0 = 0 \tag{5}$$

Whereas at time zero we set up the arbitrage portfolio at time one we unwind it. Depending on the state-of-the-world at time one our net cash inflow (a positive value) or net cash outflow (a negative value) will be...

$$\theta_b\,B_a + \theta_x\,X_a + \theta_y\,Y_a = \text{Net cash inflow/(outflow) at time one given state } \omega_a \tag{6}$$

$$\theta_b\,B_b + \theta_x\,X_b + \theta_y\,Y_b = \text{Net cash inflow/(outflow) at time one given state } \omega_b \tag{7}$$

We defined an arbitrage portfolio as having non-negative values at time one and the possibility of a positive value in one or more states at time one. Given the hypothetical problem above we will construct our portfolio such that we receive \$100 at time one regardless of the state-of-the-world at that time. Given this construct our net cash inflow at time one will be...

$$\theta_b\,B_a + \theta_x\,X_a + \theta_y\,Y_a = \$100 \text{ at time one given state } \omega_a \tag{8}$$

$$\theta_b\,B_b + \theta_x\,X_b + \theta_y\,Y_b = \$100 \text{ at time one given state } \omega_b \tag{9}$$

Given Table 1 above we know the prices of Assets B, X and Y at time zero so all we need to set up our arbitrage portfolio are the values of θ_b, θ_x and θ_y. To obtain these values we must solve Equations (5), (8) and (9) above simultaneously. The system of linear equations that we must solve is...

$$\begin{aligned} -\theta_b\,B_0 - \theta_x\,X_0 - \theta_y\,Y_0 &= 0 \\ \theta_b\,B_a + \theta_x\,X_a + \theta_y\,Y_a &= 100 \\ \theta_b\,B_b + \theta_x\,X_b + \theta_y\,Y_b &= 100 \end{aligned} \tag{10}$$

We will define matrix A as the matrix of asset prices and payoffs per Table 1 above. Noting that matrix elements in parenthesis are negative values the equation for matrix A in matrix notation is...

$$\mathbf{A} = \begin{bmatrix} (B_0) & (X_0) & (Y_0) \\ B_a & X_a & Y_a \\ B_b & X_b & Y_b \end{bmatrix} = \begin{bmatrix} (100) & (80) & (50) \\ 105 & 120 & 40 \\ 105 & 60 & 80 \end{bmatrix} \tag{11}$$

We will define vector u as the vector of arbitrage portfolio payoffs at time one. Using Equation (10) above the equation for vector u in vector notation is...

$$\vec{\mathbf{u}} = \begin{bmatrix} 0 \\ 100 \\ 100 \end{bmatrix} \tag{12}$$

We will define vector v as the vector of thetas that we must solve for. Using Equation (10) above the equation for vector v in vector notation is...

$$\vec{\mathbf{v}} = \begin{bmatrix} \theta_b \\ \theta_x \\ \theta_y \end{bmatrix} \tag{13}$$

Using Equations (11), (12) and (13) we can write our system of linear equations as a matrix:vector product. The system of linear equations that we must solve is...

$$\mathbf{A}\vec{\mathbf{v}} = \vec{\mathbf{u}} \tag{14}$$

To solve for vector v we multiply both sides of Equation (14) by the inverse of matrix A. Noting that matrix I is the identity matrix the solution to vector v is...

$$\begin{aligned} \mathbf{A}^{-1}\mathbf{A}\vec{\mathbf{v}} &= \mathbf{A}^{-1}\vec{\mathbf{u}} \\ \mathbf{I}\vec{\mathbf{v}} &= \mathbf{A}^{-1}\vec{\mathbf{u}} \\ \vec{\mathbf{v}} &= \mathbf{A}^{-1}\vec{\mathbf{u}} \end{aligned} \tag{15}$$

The Answer To Our Hypthetical Problem

Using Equation (15) above and Appendix Equation (20) below our vector v, which is the vector of thetas, is...

$$\vec{\mathbf{v}} = \mathbf{A}^{-1}\vec{\mathbf{u}} = \begin{bmatrix} (0.10435) & (0.04928) & (0.04058) \\ 0.06087 & 0.03986 & 0.01812 \\ 0.09130 & 0.03478 & 0.05217 \end{bmatrix} \begin{bmatrix} 0 \\ 100 \\ 100 \end{bmatrix} = \begin{bmatrix} (8.98551) \\ 5.79710 \\ 8.69565 \end{bmatrix} \tag{16}$$

Conclusion: If at time zero we go short 8.98551 units of Asset B, go long 5.79710 units of Asset X and go long 8.69565 units of Asset Y then this portfolio will be of zero cost to set up and will give us a guaranteed payoff of $100 in time one regardless of the state-of-the-world at that time. **We can conclude that asset prices at time zero per Table 1 above permit arbitrage**. If an arbitrage opportunity exists then demand for the assets involved would be infinte, which is inconsistent with market equilibrium.

The following tables prove that the thetas in Equation (16) above are correct...

The cost to set up the arbitrage portfolio at time zero is...

Asset	Price	Units	Position	Net CF
B	100	8.98551	Short	898.55
X	80	5.79710	Long	-463.77
Y	50	8.69565	Long	-434.78
Total				0.00

Proceeds from unwinding the portfolio at time one given state-of-the-world ω_a is...

Asset	Payoff	Units	Position	Net CF
B	105	8.98551	Short	-943.48
X	120	5.79710	Long	695.65
Y	40	8.69565	Long	347.83
Total				100.00

Proceeds from unwinding the portfolio at time one given state-of-the-world ω_b is...

Asset	Payoff	Units	Position	Net CF
B	105	8.98551	Short	-943.48
X	60	5.79710	Long	347.83
Y	80	8.69565	Long	695.65
Total				100.00

Appendix

A. The determinant of matrix A as defined by Equation (11) above is...

$$\begin{aligned} |A| &= -B_0\Bigg(X_aY_b - X_bY_a\Bigg) + X_0\Bigg(B_aY_b - B_bY_a\Bigg) - Y_0\Bigg(B_aX_b - B_bX_a\Bigg) \\ &= -100\Bigg((120)(80) - (60)(40)\Bigg) + 80\Bigg((105)(80) - (105)(40)\Bigg) - 50\Bigg((105)(60) - (105)(120)\Bigg) \\ &= -69000 \end{aligned} \tag{17}$$

Note that since the determinant of matrix A is non-zero then matrix A can be inverted. When we eliminate the arbitrage in Part II of the series this statement will be important.

B. The cofactors of matrix A as defined by Equation (11) above are...

$$\begin{aligned} a_{11} &= (-1)^{1+1}(X_aY_b - X_bY_a) = (1)[(120)(80) - (60)(40)] = 7200 \\ a_{12} &= (-1)^{1+2}(B_aY_b - B_bY_a) = (-1)[(105)(80) - (105)(40)] = -4200 \\ a_{13} &= (-1)^{1+3}(B_aX_b - B_bX_a) = (1)[(105)(60) - (105)(120)] = -6300 \\ a_{21} &= (-1)^{2+1}(X_0Y_b - X_bY_0) = (-1)[(-80)(80) - (60)(-50)] = 3400 \\ a_{22} &= (-1)^{2+2}(B_0Y_b - B_bY_0) = (1)[(-100)(80) - (105)(-50)] = -2750 \\ a_{23} &= (-1)^{2+3}(B_0X_b - B_bX_0) = (-1)[(-100)(60) - (105)(-80)] = -2400 \\ a_{31} &= (-1)^{3+1}(X_0Y_a - X_aY_0) = (1)[(-80)(40) - (120)(-50)] = 2800 \\ a_{32} &= (-1)^{3+2}(B_0Y_a - B_aY_0) = (-1)[(-100)(40) - (105)(-50)] = -1250 \\ a_{33} &= (-1)^{3+3}(B_0X_a - B_aX_0) = (1)[(-100)(120) - (105) - (80)] = -3600 \end{aligned} \tag{18}$$

C. The adjugate of matrix A using the cofactors in Appendix Equations (18) is...

$$adj(\mathbf{A}) = \begin{bmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ a_{13} & a_{23} & a_{33} \end{bmatrix} = \begin{bmatrix} 7200 & 3400 & 2800 \\ (4200) & (2750) & (1250) \\ (6300) & (2400) & (3600) \end{bmatrix} \tag{19}$$

D. Using Appendix Equations (17), (18) and (19) the inverse of matrix A as defined by Equation (11) is...

$$\begin{aligned} \mathbf{A}^{-1} &= \frac{1}{|A|}\,adj(\mathbf{A}) \\ &= -\frac{1}{69000}\begin{bmatrix} 7200 & 3400 & 2800 \\ (4200) & (2750) & (1250) \\ (6300) & (2400) & (3600) \end{bmatrix} \\ &= \begin{bmatrix} (0.10435) & (0.04928) & (0.04058) \\ 0.06087 & 0.03986 & 0.01812 \\ 0.09130 & 0.03478 & 0.05217 \end{bmatrix} \end{aligned} \tag{20}$$